The (1D) Cauchy problem to the heat equation concerns the existence and the uniqueness to the function $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ satisfying

$$\begin{aligned} u_t =& u_{xx} \qquad \text{for } (x,t) \in \mathbb{R} \times [0,T], \\ u(x,0) =& g(x) \qquad \text{for } x \in \mathbb{R}, \end{aligned}$$

where $g(x)$ is a given function.

In general, the Cauchy problem does not has a unique solution. For example, if $g(x) = 0$ then we have the following non-trivial solution.

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \varphi^{(k)}(t) x^{2k},$$

where

$$\begin{aligned} \varphi(t) &= 0 \qquad\qquad \text{for } t = 0, \\ &= e^{-t^{-2}} \qquad\qquad \text{for } t > 0. \end{aligned}$$

To obtain the uniqueness, we need to put an extra assumption that $|u(x,t)| \leqslant Ce^{ax^2}$ for some large constant C, a. See Theorem 2.12 in the textbook.

1. Uniqueness

Let us consider other conditions to guarantee the uniqueness.

Theorem 1 (Weak maximum principle)**.** *Assume that $u(x,t)$ satisfies*

$$\lim_{|x| \to +\infty} u(x,t) = 0 \tag{1}$$

and $u_t \leqslant u_{xx}$ [resp. $u_t \geqslant u_{xx}$]. Then, we have

$$u(x,t) \leqslant \sup_{x \in \mathbb{R}} g(x) \qquad [\text{ resp. } \geqslant]. \tag{2}$$

Proof. Let us consider the subsolution case. Then, one can obtain the corresponding result for the supersolutions.

Given $\epsilon > 0$, we define $w(x,t) = u(x,t) - \epsilon t - 2\epsilon$, and $w_+(x,t) = \max\{-\epsilon, w(x,t)\}$. We will show $w(x,t) \leqslant \sup g$ by contradiction. First of all, the condition (1) implies $\lim_{|x|\to+\infty} w(x,t) \leqslant -2\epsilon$. Therefore, for each $t \geqslant 0$ there exists a large number $r(t)$ such that $w_+(x,t) = -\epsilon$ holds for $|x| \geqslant r(t)$.

Given $T > 0$, we define $R = \max_{0\leqslant t\leqslant T} r(t)$. Then, we can apply the maximum principle for $Q_T = (-R,R) \times (0,T]$ so that we have

$$\max_{Q_T} w_+(x,t) \leqslant \max_{\partial_p Q_T} w_+(x,t) \leqslant \max\{-\epsilon, \sup_{\mathbb{R}} g(x)\} = \sup_{\mathbb{R}} g(x).$$

On the other hand,

$$-\epsilon T - 2\epsilon + \sup_{x\in\mathbb{R}, t\in[0,T]} u(x,t) \leqslant \sup_{x\in\mathbb{R}, t\in[0,T]} w(x,t) \leqslant \max_{Q_T} w_+(x,t) \leqslant \sup_{\mathbb{R}} g(x).$$

Hence, by passing ϵ to 0, we have

$$\sup_{x\in\mathbb{R}, t\in[0,T]} u(x,t) \leqslant \sup_{\mathbb{R}} g(x),$$

for any $T > 0$. Therefore, $u(x,t) \leqslant \sup g$.

□

Then, the maximum principle directly implies the following uniqueness theorem.

Theorem 2 (Uniqueness)**.** *Given a smooth function $g : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{|x|\to+\infty} g(x) = 0$, the heat equation has a unique solution satisfying $\lim_{|x|\to+\infty} u(x,t) = 0$.*

On the other hand, we can also have the following monotonicity.

Theorem 3 (Monotonicity)**.** *A solution $u(x,t)$ to the heat equation on $\mathbb{R} \times [0,T]$ with bounded L^2-norm satisfies*

$$\int_{\mathbb{R}} u^2(x,t)dx \leqslant \int_{\mathbb{R}} u^2(x,0)dx.$$

The bonded L^2 norm implies that given $T > 0$ there exists a constant $C(T)$ such that

$$\int_{\mathbb{R}} u^2(x,t)dx \leqslant C(T),$$

holds for $t \in [0,T]$.

Proof. Given large $R > 0$, we define a *cut-off* function $\eta_R(x) = (1 - x^2R^{-2})_+ = \max\{0, 1 - x^2R^{-2}\}$. Then,

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\eta_R^2 u^2 dx = \int_{-R}^{R} \eta_R^2 u u_t dx = \int_{-R}^{R} -\eta_R^2 u_x^2 - 2\eta_R(\eta_R)_x u u_x dx \leqslant \int_{-R}^{R} |\partial_x \eta_R|^2 u^2 dx \leqslant \frac{4}{R^2}\int_{\mathbb{R}} u^2 dx.$$

Thus,

$$\frac{1}{2}\int \eta_R^2 u^2(x,t)dx - \frac{1}{2}\int \eta_R^2 u^2(x,0)dx \leqslant \frac{4}{R^2}\int_0^t \int u^2(x,s)dxds \leqslant \frac{4tC}{R^2},$$

for some constant C. Since $\eta_R(x)$ monotonically converges to 1 as $R \to +\infty$, we have

$$\frac{1}{2}\int u^2(x,t)dx - \frac{1}{2}\int u^2(x,0)dx \leqslant \lim_{R\to+\infty} \frac{4tC}{R^2} = 0.$$

□

Thus, we can obtain the following uniqueness theorem.

Theorem 4 (Uniqueness)**.** *Given a smooth function $g : \mathbb{R} \to \mathbb{R}$ with bounded L^2-norm, the heat equation has a unique solution with bounded L^2-norm.*

2. Existence

Now, we construct a solution to show the existence. Given continuous $g(x)$ satisfying $|g(x)| \leqslant Ce^{ax^2}$ for some constant C, a, we have a solution

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} g(y)dy,$$

where $t > 0$. It is easy to check whether the function above satisfies $u_t = u_{xx}$. So, we only need to check $u(x,t) \to g(x)$ as $t \to 0^+$.

For simplicity, we only consider the case that $|g(x)| \leqslant M$ and that $g(x)$ is continuous.

$$g(x) = g(x)\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} dy = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} g(x)dy.$$

Hence,

$$|u(x,t) - g(x)| \leqslant \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} |g(y) - g(x)|dy.$$

Given small $\epsilon > 0$, there exits $\delta > 0$ such that $|g(y) - g(x)| \leqslant \epsilon$ holds for $|y - x| \leqslant \delta$.

$$
\begin{aligned}
|u(x,t) - g(x)| &\leqslant \int_{|x-y|\leqslant\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} |g(y) - g(x)| dy + \int_{|x-y|>\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} |g(y) - g(x)| dy \\
&\leqslant \epsilon \int_{|x-y|\leqslant\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} dy + 2M \int_{|x-y|>\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} dy.
\end{aligned}
$$

For a fixed $t > 0$, we substitute $(x - y)/(4t)^{-\frac{1}{2}}$ by z so that

$$
|u(x,t) - g(x)| \leqslant \frac{\epsilon}{\sqrt{\pi}} \int_{|z|\leqslant\delta\sqrt{4t}} e^{-z^2} dz + \frac{2M}{\sqrt{\pi}} \int_{|z|>\delta\sqrt{4t}} e^{-z^2} dz \leqslant \frac{\epsilon}{\sqrt{\pi}} + \frac{2M}{\sqrt{\pi}} \int_{|z|>\delta\sqrt{4t}} e^{-z^2} dz.
$$

Next, given small $\epsilon > 0$ there exists a small enough T such that for $t \in (0, T)$ the following holds

$$
\int_{|z|>\delta\sqrt{4t}} e^{-z^2} dz \leqslant \epsilon.
$$

Hence, for $t \in (0, T)$ we have

$$
|u(x,t) - g(x)| \leqslant (2M + 1)\pi^{-\frac{1}{2}} \epsilon.
$$

Therefore, as $t \to 0^+$, we have $u(x,t) \to g(x)$.

We also notice that given $g(x)$, the equation $u_t(x,t) = u_{xx}(x,t) + f(x,t)$ has a solution

$$
u(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} g(y) dy + \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s) dy ds.
$$